# THE DOUBLE TANGENT BUNDLE

KEATON QUINN

Abstract. The following are notes to myself on the double tangent bundle to a smooth manifold. There are currently no citations included but I make no claim that anything listed here is my original work.

## 1. Introduction

**Theorem.** *Suppose $f : M \to N$ is a smooth map. Then $df : TM \to TN$ is also a smooth map with derivative*

$$d(df)_{(p,v)}(x, y) = (df_p(x), df_p(y) + \nabla(df)_p(v, x)).$$

In the case that $N = \mathbb{R}$, the total covariant derivative $\nabla(df)$ reduces to the Hessian of the function $f$.

**Corollary.** *If $f : M \to \mathbb{R}$ is smooth function then*

$$df_{(p,v)}(x, y) = (df_p(x), df_p(y) + \mathrm{Hess}(f)_p(v, x)).$$

Say $f : \Sigma \to M$ is an immersion and $V$ is a vector field along on $\Sigma$ with values in $TM$, that is, $V$ is a section of $f^*TM$. Then a map $F : \Sigma \to TM$ can be made by $F(x) = (f(x), V(x))$ and its derivative is related to the pullback of the connection on $M$.

**Theorem.** *The derivative $dF_x : T_x\Sigma \to T_{(f(x),V(x))}(TM)$ is given by*

$$dF_x(u) = (df_x(u), (f^*\nabla)_u V(x)).$$

This can be made more precise when $M$ carries a Riemannian metric and the vector field $V$ is normal to the the surface $\Sigma$.

**Corollary.** *Suppose $f : \Sigma \to M$ is an isometric immersion and suppose $\Sigma$ has a unit normal vector field $N$. Then the map $F(x) = (f(x), N(x))$ gives a map $F : \Sigma \to TM$ and its derivative at a point $x$ is*

$$dF_x(u) = (df_x(u), -df_x(Bu)).$$

Several constructions related to the tangent bundle of a Riemannian manifold preserve the norms of vectors, and so if a vector starts out with unit length, it will stay unit length. It is natural, then, to consider the unit tangent bundle $UM \to M$ of all unit vectors tangent to the manifold. As $UM$ sits inside the tangent bundle as an embedded submanifold, the tangent spaces to $UM$ inherit a splitting from the connection on $TM$.

Last Revised: August 19, 2022.

**Lemma.** *The tangent space to $UM$ at $(p,v)$ is given by*

$$T_{(p,v)}UM = H_{(p,v)}(TM) \oplus v^{\perp}$$

*and the tangent space to $TM$ can be written as*

$$T_{(p,v)}TM = H_{(p,v)}TM \oplus v^{\perp} \oplus \langle v \rangle.$$

*where $\langle v \rangle$ is the span of $v$ in $T_pM$.*

The exponential map $\exp : TM \to M$ sends a point $(p,v)$ to the point in $M$ a unit distance away from $p$ along the geodesic emanating from $p$ in the direction $v$. Its derivative is related to the variation field of a variation through geodesics.

**Theorem.** *The derivative of the exponential map is given by*

$$d\exp_{(p,v)}(x,y) = J(1)$$

*where $J(t)$ is the unique Jacobi field along $\exp_p(tv)$ with $J(0) = x$ and $D_tJ(0) = y$.*

The geodesic flow is the map $g^t : UM \to UM$ that is the flow of the geodesic vector field $G : UM \to T(UM)$ given by $G(p,v) = (v,0)$. This flow is given in terms of the exponential map by $g^t(p,v) = (\exp_p(tv), \partial_t \exp_p(tv))$ and its derivative can also be computed using Jacobi fields.

**Theorem.** *The derivative of the geodesic flow $g^t : TM \to TM$ is given by*

$$d(g^t)_{(p,v)}(x,y) = (J(t), D_tJ(t)).$$

*where $J(t)$ is the unique Jacobi field along $\exp_p(tv)$ with $J(0) = x$ and $D_tJ(0) = y$.*

Returning to the unit normal lift $F : \Sigma \to UM$, we can compose this with the geodesic flow to get surfaces parallel to the original $\Sigma$. The derivative of this composition is computed using the chain rule.

**Theorem.** *Let $F_t = g^t \circ F : \Sigma \to UM$. Then the derivative is*

$$d(F_t)_x(u) = (J(t), D_tJ(t))$$

*where $J(t)$ is the unique Jacobi field along $\exp_{f(x)}(tN(x))$ with $J(0) = df_x(u)$ and $D_tJ(0) = -df_x(Bu))$ for $B$ the shape operator of $\Sigma$ in $M$. If $f_t = \pi \circ F_t$, then*

$$d(f_t)_x(u) = J(t).$$

Using this derivative, the induced metric $I_t = f_t^*g$ can be computed in terms of Jacobi fields and the time derivative of $I_t$ at $t = 0$ is related to the second fundamental form of the immersion $f$.

**Theorem.** *Let $f : \Sigma \to M$ be an immersed hypersurface and let $N$ be a unit normal vector field on $\Sigma$. Then for the induced metric on the parallel surface a distance $t$ away from $\Sigma$ in the direction $N$ we have*

$$\frac{d}{dt}\, I_t\big|_{t=0} = -2I\!I.$$

*where $I\!I$ is the second fundamental form of the immersion $f$ with respect to $N$.*

The tangent bundle $TM$ (and hence $UM$) inherits a Riemannian metric from $(M,g)$ called the Sasaki metric $\hat{g}$. The pullback of this metric via the unit normal immersion $F : \Sigma \to UM$ is related to the geometry of $f : \Sigma \to M$.

**Theorem.** *Let $F : \Sigma \to UM$ by $F(x) = (f(x), N(x))$ be a unit normal immersion into the unit tangent bundle of a manifold with metric $g$. Let $\hat{g}$ be the Sasaki metric on $UM$ and call the induced metric $\hat{I} = F^*\hat{g}$. Then*

$$\hat{I} = I + I\!I\!I$$

*where $I$ is the induced metric $f^*g$ on $\Sigma$ and $I\!I\!I$ is the third fundamental form.*

1.1. **Hyperbolic Space.** When $M = \mathbb{H}^n$, the above formulas can be made more explicit.

**Theorem.** *Let $g^t : U\mathbb{H}^n \to U\mathbb{H}^n$ be the geodesic flow. The derivative is given by*

$$d(g^t)_{(p,v)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh(t)x + \sinh(t)y + \sinh(t)\langle x, v\rangle p \\ \sinh(t)x + \cosh(t)y - \sinh(t)\langle x, v\rangle v \end{pmatrix},$$

*with respect to the splitting $T(T\mathbb{H}^n) = H \oplus V$.*

From the model of hyperbolic space as the hyperboloid in Minkowski space, there is a map $U\mathbb{H}^n \to TdS^n$ from the unit tangent bundle of hyperbolic space to the tangent bundle of de Sitter space. The map is simply $(p, v) \mapsto (v, p)$, which uses the fact that each are submanifolds of Minkowski space, and so tangent vectors can be identified with actual vectors in Minkowski space.

**Theorem.** *Let $i : U\mathbb{H}^n \to TdS^n$ be given by $i(p, v) = (v, p)$. Then with respect to the splitting of the double tangent bundle of both $\mathbb{H}^n$ and $dS^n$ into horizontal and vertical bundles (via the metrics induced from Minkowski space), we can write the derivative as*

$$di_{(p,v)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + \langle x, v\rangle p \\ x - \langle x, v\rangle v \end{pmatrix}$$

Via this map, the unit tangent bundle of hyperbolic space inherits a metric from the Sasaki metric of de Sitter space.

**Lemma.** *Let $\hat{g}_{dS}$ be the Sasaki metric on $TdS^n$. Then*

$$\begin{aligned} i^*(\hat{g}_{dS})((x, y), (u, w)) &= \langle x, y\rangle - 2\langle x, v\rangle\langle u, v\rangle + \langle y, w\rangle \\ &= \hat{g}_{\mathbb{H}}((x, y), (u, w)) - 2\langle x, v\rangle\langle u, v\rangle, \end{aligned}$$

*where $\hat{g}_{\mathbb{H}}$ is the Sasaki metric on $U\mathbb{H}^n$.*

Also from this map, we get dual surfaces in de Sitter space from unit normal immersions in hyperbolic space. If $F : \Sigma \to U\mathbb{H}^n$ is a unit normal immersion, then $F^* = i \circ F$ and $f^* = \pi \circ F^*$ give maps $\Sigma \to TdS^n$ and $\Sigma \to dS^n$, respectively. The geometry of these dual surfaces is related to the originals.

**Lemma.** *The derivative of $F^*$ is given by*

$$dF^*_x(u) = \begin{pmatrix} -df_x(Bu) \\ df_x(u) \end{pmatrix},$$

*and the derivative of $f^*$ is*

$$df^*_x(u) = -df_x(Bu).$$

With these derivatives the induced metrics can be computed.

**Theorem.** *Let $f : \Sigma \to \mathbb{H}^n$ be a map and $F : \Sigma \to U\mathbb{H}^n$ via $F(x) = (f(x), N(x))$ be a unit normal immersion. Then the metrics $I^*$ and $\hat{I}^*$ on $\Sigma$ obtained by pulling back the metric on $dS^n$ or the Sasaki metric on $TdS^n$, respectively, are given by*

$$\hat{I}^* = I + I\!I\!I,$$

*and*

$$I^* = I\!I\!I$$

From a unit normal immersion into hyperbolic space we get parallel copies of the surface as described above. In the hyperbolic setting, the there is an explicit description of the metrics $I_t$.

**Theorem.** *Let $f : \Sigma \to \mathbb{H}^n$ be an immersion with unit normal vector field $N$ and shape operator $B$. Define $f_t(x) = \pi \circ g^t(f(x), N(x))$, then the image of $f_t$ is a surface a distance $t$ away from $\Sigma$ and has in induced metric*

$$I_t(u, w) = I((\cosh(t)Id - \sinh(t)B)u, (\cosh(t)Id - \sinh(t)B)w)$$

*and shape operator*

$$B_t = -(\cosh(t)Id - \sinh(t)B)^{-1}(\sinh(t)Id + \cosh(t)B).$$

## 2. Structure of the double tangent bundle

Let $M$ be a smooth manifold and let $\pi : TM \to M$ be its tangent bundle. The total space $TM$ is itself a manifold twice the dimension of $M$. It is defined as the (disjoint) union of the tangent spaces to $M$, i.e,

$$TM = \bigcup_{p \in M} T_pM.$$

An element of $TM$ looks like $(p, v)$ where $p \in M$ and $v \in T_pM$. The projection $\pi : TM \to M$ is $\pi(p, v) = p$. The tangent bundle records derivatives of maps $\mathbb{R} \to M$ (as opposed to the cotangent bundle that records derivatives $M \to \mathbb{R}$). If $\gamma : I \to M$ is a smooth map through the point $p \in M$, then $d\gamma_0(\partial_t) = \gamma'(0) \in T_pM$.

But what about second derivatives? That is, suppose $\gamma : I \to M$ is a smooth path through $p$. What is $\frac{d^2}{dt^2}\gamma(0)$? For each $t$, the derivative $\gamma'(t)$ lives in a different tangent space. So, to consider this as a path in a smooth manifold (so that we may take another derivative), we think of it as as $(\gamma(t), \gamma'(t))$, which is a smooth path in $TM$. We then interpret the second derivative of $\gamma$ as $\frac{d}{dt}(\gamma(t), \gamma'(t))$ at $t = 0$. This is a tangent vector to $TM$ at $(p, \gamma'(0))$, i.e., an element of $T(TM)$, the tangent bundle of the tangent bundle of $M$.

If we start at a point $(p, v)$ in $TM$ and try moving, there are two distinguished ways to do so. We could move along the base manifold $M$ or move within the fibers of $\pi : TM \to M$, meaning move within the tangent spaces. Usually both motions will happen at the same time. So we expect $T_{(p,v)}TM$ to decompose, in some sense, into directions tangent to the base $M$ and directions tangent to the fibers.

To find the tangent vectors in the direction of the fibers take a path $\gamma(t) = (p, v(t))$ completely within $T_pM \subset TM$ starting at $(p, v)$. Then since $\pi(\gamma(t)) = p$ for all $t$, we see $d\pi_{(p,v)}(\gamma'(t)) = 0$, or that $\gamma'(t)$ is in the kernel of $d\pi$. And indeed, the tangent space to the fiber $T_pM$ is the kernel of $d\pi_{(p,v)}$. Since $\pi$ is a submersion (given $v \in T_pM$, the tangent vector to $(\gamma(t), \gamma'(t))$ maps to $v$ for any curve $\gamma$ in $M$ with $\gamma'(0) = v$), by the regular value theorem we have

$$\ker d\pi_{(p,v)} = T_{(p,v)}\pi^{-1}(p) = T_{(p,v)}T_pM.$$

We define the *vertical tangent space* at $(p, v) \in TM$ to be

$$V_{(p,v)} = \ker d\pi_{(p,v)} = T_{(p,v)}T_pM \simeq T_pM.$$

It consists of all directions tangent to the fiber and is canonically isomorphic to the fiber itself (since the tangent spaces to a vector space are canonically isomorphic to the vector space). Note that this vertical space can be defined for any smooth manifold and does not depend on the geometry in any way.

We expect that the remaining vectors in $T_{(p,v)}TM$ should represent motion along the base, and indeed we have

$$T_{(p,v)}TM/V_{(p,v)} \simeq \operatorname{im} d\pi_{(p,v)} = T_pM.$$

This information can be packaged together into the short exact sequence of vector spaces

$$0 \to V_{(p,v)} \to T_{(p,v)}TM \to T_pM \to 0,$$

which comes from the short exact sequence of vector bundles on $TM$

$$0 \to V \to T(TM) \to \pi^*TM \to 0.$$

However, this sequence does not canonically split.

A connection on $TM$ is a choice of splitting of $T(TM) = H \oplus V$ for a smooth family of subspaces $H$ complementary to $V$ in the sense that $T_{(p,v)}TM = H_{(p,v)} \oplus V_{(p,v)}$. This $H_{(p,v)}$ is called the *horizontal tangent space* at $(p, v)$, and indeed this space represents motions in $TM$ along $M$ since

$$H_{(p,v)} \simeq T_{(p,v)}TM/V_{(p,v)} \simeq T_pM,$$

the isomorphism $H \to TM$ given by $d\pi$. The connection is called linear if it is invariant under scalar multiplication on the Tangent bundle. That is, let $r \in \mathbb{R}$ and $L_r : TM \to TM$ be given by $L_r(p, v) = (p, rv)$. Then the connection is linear if

$$H_{(p,rv)} = d(L_r)_{(p,v)}(H_{(p,v)}).$$

A right splitting $j : \pi^*TM \to T(TM)$ of the short exact sequence $0 \to V \to T(TM) \to \pi^*TM \to 0$ let's us write $T_{(p,v)}TM = j_{(p,v)}(T_pM) \oplus V_{(p,v)}$, and so $H = j(\pi^*TM)$ defines a connection. A left splitting $k : T(TM) \to V$ is a projection onto the vertical space, and a connection can be defined as $H = \ker k$. Note that either splitting gives the other, and the resulting connection will be linear if the splitting maps are invariant under scalar multiplication (CHECK THIS). And to reiterate, there are many connections on $TM$, but no canonical one without introducing more structure on $M$.

Now, suppose we have chosen a linear connection on $TM$. Then since $V_{(p,v)} \simeq T_pM$ (canonically) and since $H_{(p,v)} \simeq T_pM$ by $d\pi_{(p,v)}$ we have that

$$T_{(p,v)}TM \simeq T_pM \oplus T_pM,$$

given by the isomorphism that sends $W \in T_{(p,v)}TM$ to $(d\pi_{(p,v)}(W), k(W)) \in T_pM \oplus T_pM$, where $k$ is the corresponding left splitting given by the connection.

## 3. Covariant Derivatives

3.1. **Covariant derivatives.** Take a curve $\alpha : (-1,1) \to TM$ by $\alpha(t) = (\gamma(t), V(t))$ starting at $(p, v)$. Then $\alpha'(0) = d\alpha_0(\partial_t) \in T_{(p,v)}TM \simeq T_pM \oplus T_pM$. The first component of this tangent vector is given by

$$d\pi_{(p,v)}(\alpha'(0)) = \frac{d}{dt}\,\pi(\alpha(t))|_{t=0} = \frac{d}{dt}\,\gamma(t)|_{t=0} = \gamma'(0).$$

So we see explicitly that the horizontal component of a tangent vector to $TM$ represents motion along the base. The second component is given by

$$k(\alpha'(0)) = k \circ d\alpha_0(\partial_t)$$

Therefore, using $k$ we can talk about derivatives of tangent vectors on $M$.

Let $Y : M \to TM$ be a vector field so that $\pi \circ Y = Id$. Then its derivative is a map $dY : TM \to T(TM)$. Using the decomposition of the double tangent bundle, the derivative of $Y$ at $p \in M$ in the direction $x \in T_pM$ can be computed by

$$dY_p(x) = (x, k \circ dY(x)).$$

So $k \circ dY$ carries all the information about the derivative of $Y$. The covariant derivative of the vector field $Y$ with respect to (or in the direction of) the vector field $X$ is another vector field $\nabla_X Y$ given by

$$\nabla_X Y(p) = k \circ dY_p(X_p).$$

A connection on $TM$ more frequently refers to this $\nabla$ than to the splitting $H$.

3.2. **Covariant derivatives along curves.** (ADD EXPOSITION ABOUT VECTOR FIELDS ALONG CURVES) Let $\gamma : (-1,1) \to M$ be a curve and consider the pullback bundle $\gamma^*TM$ whose fiber over $t \in (-1,1)$ is the vector space $T_{\gamma(t)}M$. Put another way

$$\gamma^*TM = \{(t, v) \mid \gamma(t) = \pi(v)\} \subset (-1,1) \times TM.$$

There is a natural pushforward $\gamma_* : \gamma^*TM \to TM$ (note: not the derivative) given by $\gamma_*(t, v) = (\gamma(t), v)$. Since $\gamma^*TM \subset (-1,1) \times TM$, we have that $T_{(t,v)}\gamma^*TM \leq T_t(-1,1) \oplus H_{(\gamma(t),v)} \oplus V_{(\gamma(t),v)}$. If $(t(s), V(s))$ is a path in this pullback bundle starting at $(t, v)$ with an initial velocity $(a\partial_t, x, y)$, then from the condition $\gamma(t(s)) = \pi(V(s))$ and taking an $s$-derivative at $s = 0$, we see that $a\gamma'(t) = d\gamma_t(a\partial_t) = d\gamma_t(t'(0)) = d\pi_{(\gamma(t),v)}(x, y) = x$. Consequently, the tangent space is identified as

$$T_{(t,v)}\gamma^*TM = \{(a\partial_t, a\gamma'(t), y) \mid a \in \mathbb{R} \text{ and } y \in T_{\gamma(t)}M\}.$$

Since $\gamma_*(t, v) = (\gamma(t), v)$, its derivative is $d(\gamma_*)_{(t,v)}(a\partial_t, a\gamma'(t), y) = (a\gamma'(t), y)$.

The bundle projection $\pi_1 : \gamma^*TM \to (-1,1)$ is just projection onto the first component: $\pi_1(t, v) = t$, and so its derivative is $d(\pi_1)_{(t,v)}(a\partial_t, a\gamma'(t), y) = a\partial_t$. The vertical bundle of $\gamma^*TM$ is then given by

$$V^*_{(t,v)} = \ker d(\pi_1)_{(t,v)} = \{(0, 0, y)\}.$$

and note that $d\gamma_*$ sends $(0, 0, y) \mapsto (0, y)$, preserving vertical vectors, i.e, it maps $V^*$ to $V$. If $k^* : T(\gamma^*TM) \to V^*$ is the corresponding projection to the vertical space, then we see

$$d\gamma_* \circ k^* = k \circ d\gamma_*.$$

and therefore the induced connection $\{a(\partial_t, \gamma'(t), 0)\}$ is mapped to the horizontal space $H$ by $d\gamma_*$.

From this induced connection is an induced covariant derivative $D_t = \gamma^*\nabla$ that acts on section of $\gamma^*TM$. If $V \in \Gamma(\gamma^*TM)$ is a vector field along $\gamma$, then the covariant derivative of $V$ along $\gamma$ is

$$D_tV = (\gamma^*\nabla)_{\partial_t}V = k^* \circ dV(\partial t).$$

While $D_tV$ is another section of $\gamma^*TM$ we will frequently use the same notation to also refer to $d\gamma_* \circ D_tV$ in $TM$.

If $V$ is extendible, meaning if $V(t) = (t, V_t)$ and $V_t = \tilde{V}_{\gamma(t)} \in T_{\gamma(t)}M$ for a vector field $\tilde{V}$ on $M$, then $D_tV$ is related to $\nabla_{\gamma'(t)}\tilde{V}$. Indeed

**Lemma.** *If $V$ is an extendible vector field along the curve $\gamma$, then for every extension $\tilde{V}$ of $V$,*

$$D_tV = \nabla_{\gamma'(t)}\tilde{V}.$$

*after identifying the fibers of the pullback bundle with the fibers of $TM$ via $d\gamma_*$.*

*Proof.* If $\tilde{V}$ is an extension of $V$ and if $V(t) = (t, V_t)$ and $\tilde{V}(t) = (\gamma(t), \tilde{V}_\gamma(t))$, then $\gamma_* \circ V = \tilde{V} \circ \gamma$. This implies that

$$d(\gamma_*)_{V(t)} \circ dV_t = d\tilde{V}_{\gamma(t)} \circ d\gamma_t.$$

Together with $d\gamma_* \circ k^* = k \circ d\gamma_*$ we compute

$$\begin{aligned} d(\gamma_*)_{V(t)} \circ D_tV &= d(\gamma_*)_{V(t)} \circ k^* \circ dV_t(\partial_t) \\ &= k \circ d(\gamma_*)_{V(t)} \circ DV_t(\partial_t) \\ &= k \circ d\tilde{V}_{\gamma(t)} \circ d\gamma_t(\partial_t) \\ &= k \circ d\tilde{V}_{\gamma(t)}(\gamma'(t)) \\ &= \nabla_{\gamma'(t)}\tilde{V}. \end{aligned}$$

□

Using this induced covariant derivative we can now compute the derivative of the curve $\alpha(t) = (\gamma(t), V(t))$ in $T(TM)$.

**Theorem.** *Let $\alpha : (-1, 1) \to TM$ be a smooth path given by $\alpha(t) = (\gamma(t), V(t))$ and starting at $(p, v)$. Then*

$$\alpha'(0) = (\gamma'(0), D_tV(0)) \in T_{(p,v)}TM$$

*Proof.* We've already seen the horizontal component is given by $\gamma'(0)$ and the vertical component is $k \circ d\alpha_0(\partial_t)$. If we interpret $V$ as a section of $\gamma^*TM$ given by (abusing notation) $V(t) = (t, V(t))$, then $\gamma_* \circ V = \alpha$. So,

$$\begin{aligned} k \circ d\alpha_0(\partial_t) &= k \circ d(\gamma_*)_{(0,v)} \circ dV_0(\partial_t) \\ &= d(\gamma_*)_{(0,v)} \circ k^* \circ dV_0(\partial_t) = d(\gamma_*)_{(0,v)}(D_tV(0)) \simeq D_tV(0), \end{aligned}$$

as claimed. □

To expand on this, if $\alpha(t) = (\gamma(t), V(t))$ is a curve in $TM$ with $\alpha(0) = (p, v)$ and $\alpha'(0) = (x, y) \in H \oplus V$, then $\gamma'(0) = x$ and $D_tV(0) = y$. This helps us compute the derivative of functions defined on $TM$. Suppose $F : TM \to N$ is a smooth map and we want to compute $dF_{(p,v)}(x, y)$. Then take a curve $\alpha(t) = (\gamma(t), V(t))$ starting at $(p, v)$ with $\gamma'(0) = x$ and $D_tV(0) = y$. Then

$$dF_{(p,v)}(x, y) = \frac{d}{dt}\, F(\alpha(t))\big|_{t=0} = \frac{d}{dt}\, F(\gamma(t), V(t))\big|_{t=0}.$$

3.3. **Parallel transport.** Let $\gamma$ be a curve in $M$. If we wanted to compute $\gamma''(t)$ using a difference quotient we would need to compute the difference $\gamma'(t+h)-\gamma'(t)$, but these two vectors live in different tangent spaces. If we had a way to move $\gamma'(t+h)$ back into $T_{\gamma(t)}M$ then we could perform the subtraction. Moving $\gamma'(t+h)$ is moving a vector within $TM$, and to get an 'accurate' representation of this vector in $T_{\gamma(t)}M$ we don't want any of this motion to happen in the fiber directions. So, if for $s \in [0,1]$ we have $(\gamma(s), V(s))$ is the desired path in $TM$ such that $V(0) = \gamma'(t)$ and $V(1) = \gamma'(t+h)$ then we want this be horizontal. That is, we want $D_s V = 0$ for all $s \in [0,1]$. Such a path is called a parallel transport of $\gamma'(t)$ along $\gamma$. The following theorem shows that given any $v \in T_P M$ and $\gamma : [0,1] \to M$ a curve starting at $p$, then we can parallel transport $v$ along $\gamma$.

**Theorem** (Parallel Transport)**.** *Let $p$ and $q$ be two points in $M$ and suppose $\gamma : [0,1] \to M$ is a path starting at $p$ and ending at $q$. Then for any $v \in T_p M$ there exists a unique vector field $V$ along $\gamma$ such that $V(0) = v$ and $D_t V = 0$ for all $t$.*

*Proof.* Again, consider the pullback bundle $\gamma^* TM$. Define the parallel transport vector field $P : \gamma^* TM \to T(\gamma^* TM)$ by $P(t,w) = (\partial_t, \gamma'(t), 0)$ and note this is a horizontal vector field. Let $\alpha(s) = (t(s), V(s))$ be an integral curve of this vector field through the point $(t,v)$. Then we have

$$(t'(s)\partial_t, t'(s)\gamma'(t(s)), D_s V(s)) = \alpha'(s) = P(\alpha(s)) = (\partial_t, \gamma'(t(s)), 0)$$

so that $t(s) = t + s$. Hence $\alpha(s) = (t+s, V(s))$ where $D_s V = 0$ and $V(0) = v$. This is the unique integral curve of $P$ through $(t,v)$ defined on some small interval around $s = t$, i.e., on $(t - \epsilon_t, t + \epsilon_t)$ for $\epsilon_t > 0$.

For reasons I haven't quite worked out yet, the integral curve starting at $(0,v)$ exists for all $s \in [0,1]$. Another way to phrase this is that $P$ is a complete vector field. Should be able to show $P$ is a system of linear ODEs in charts and then just apply the ODE theorem as usual. $\square$

Parallel transport can be used to connect nearby tangent spaces.

**Theorem.** *Let $\gamma$ be a curve in $M$, then for each $t$ and $s$ in the domain of $\gamma$, there exists a linear isometry $P(\gamma)_t^s : T_{\gamma(t)}M \to T_{\gamma(s)}M$ given by*

$$P(\gamma)_t^s(v) = V(1),$$

*where $V$ is the parallel transport of $v$ along $\gamma$ restricted to the interval with endpoints $t$ and $s$.*

Note that $P(\gamma)_s^t$ is the inverse of $P(\gamma)_t^s$. This parallel transport map is used to compute the derivative of tangential objects. For example,

**Theorem.** *Let $\gamma$ be a curve in $M$, then $\frac{d^2}{d^2}\gamma(t) = (\gamma'(t), D_t\gamma'(t))$ and*

$$D_t\gamma'(t) = \lim_{h\to 0} \frac{P(\gamma)_{t+h}^t(\gamma'(t+h)) - \gamma'(t)}{h}.$$

3.4. **Derivatives of functions.** Suppose $f : M \to N$ is a smooth map between manifolds. Then its differential $df : TM \to TN$ is given by $df(p,v) = (f(p), df_p(v))$. We therefore have another derivative $d(df) : T(TM) \to T(TN)$, the derivative of the differential (note: this is not the exterior derivative applied twice, which would be zero). The differential $df$ more naturally takes values in the pullback bundle $f^*TN$, meaning itself is a section of the bundle $T^*M \otimes f^*TN \simeq \mathrm{Hom}(TM, TN)$

over $M$. Suppose $M$ has a connection $\nabla^M$ and $N$ has a connection $\nabla^N$, then the bundle $\mathrm{Hom}(TM, TN)$ has the induced connection $\nabla = (\nabla^M)^* \otimes \nabla N$, and with connection we can compute the derivative of $df$.

**Theorem.** *Suppose $f : M \to N$ is a smooth map. Then*

$$d(df)_{(p,v)}(x, y) = (df_p(x), df_p(y) + \nabla(df)_p(v, x)).$$

*Proof.* Let $\alpha(t) = (\gamma(t), V(t)$ be a curve in $TM$ such that $\alpha(0) = (p, v)$ and $\alpha'(0) = (x, y)$. Recall this means $\gamma'(0) = x$ and $D_t V(0) = y$. Now we can compute

$$\begin{aligned} d(df)_{(p,v)}(x, y) &= \frac{d}{dt}\, df(\gamma(t), V(t))\big|_{t=0} \\ &= \frac{d}{dt}\, (f(\gamma(t)), df_{\gamma(t)}(V(t)))\big|_{t=0} \\ &= (df_p(\gamma'(0)), D_{t'}(df_{\gamma(t)}(V))(0)) \end{aligned}$$

where $D_{t'}(df_{\gamma(t)}(V))$ is the covariant derivative of the vector field $df_{\gamma(t)}(V(t))$ along the curve $f \circ \gamma$.

By definition,

$$\nabla(df)(Y, X) = (\nabla_X df)(Y) = (f^*\nabla)_X df(Y) - df(\nabla_X Y),$$

which implies that

$$(f^*\nabla)_X df(Y) = df(\nabla_X Y) + \nabla(df)(Y, X).$$

Now take $X = \gamma'$, then $\nabla_X Y = D_t Y$ and since $Y$ is defined on all of $M$, we have

$$\begin{aligned} (f^*\nabla)_{\gamma'} df(Y) &= \gamma^*(f^*\nabla)_{\partial_t}(df(Y) \circ \gamma) \\ &= (f \circ \gamma)^* \nabla_{\partial_t}(df(Y) \circ \gamma) = D_{t'} df(Y), \end{aligned}$$

i.e., the covariant derivative of $df(Y)$ along $f \circ \gamma$. Now take $Y = V$ along $\gamma$, we get

$$D_{t'} df(V) = df_{\gamma(t)}(D_t V) + \nabla(df)(v, \gamma').$$

At $t = 0$ this reads

$$D_{t'} df(V)(0) = df_p(y) + \nabla(df)(v, x),$$

which gives the result. □

The total covariant derivative $\nabla(df)$ is also called the second fundamental form of $f$, and indeed, if $f : M \to N$ is an isometric immersion of Riemannian manifolds, then $\nabla(df) = I\!I$. This follows from the Gauss Formula

$$(f^*\nabla)_X df(Y) = df(\nabla_X Y) + I\!I(X, Y).$$

If $f$ is a totally geodesics mapping of Riemannian manifolds, then $\nabla(df) = 0$. To see this, take any $(p, v) \in TM$ and geodesic at $p$ in the direction $v$, then the covariant derivative of $df(\gamma')$ along $f \circ \gamma$ is $D_{t'}(f \circ \gamma)' = 0$, since $f \circ \gamma$ is also a geodesic. Then

$$0 = D_{t'}(f \circ \gamma)' = df(D_t \gamma') + \nabla(df)(\gamma', \gamma') \implies \nabla(df)(\gamma', \gamma') = 0.$$

At $t = 0$ this gives $\nabla(df)(v, v) = 0$ and, since $v$ was arbitrary, $\nabla(df) = 0$. In particular, any isometry $f : M \to M$ also has $\nabla(df) = 0$ since it is totally geodesic. This also shows any totally geodesic map is a harmonic map since $\mathrm{tr}(\nabla(df)) = \mathrm{tr}(0) = 0$. So all isometries of a Riemannian manifold are harmonic.

In the special case of $N = \mathbb{R}$ the total covariant derivative of $df$ is called the Hessian, so we have the following.

**Corollary.** *If $f : M \to \mathbb{R}$ is smooth function then*

$$df_{(p,v)}(x, y) = (df_p(x), df_p(y) + \mathrm{Hess}(f)_p(v, x)).$$

From inspection, we can interpret this as saying the Hessian of a smooth function measures the change in $df$ along the base manifold and since $df_p$ is linear, the change of $df$ in the fiber directions is just itself.

Now suppose $f : \Sigma \to M$ is a hypersurface and $V$ is a vector field along $\Sigma$. Then $F(x) = (f(x), V(x))$ gives a map into the tangent bundle $F : \Sigma \to TM$.

**Theorem.** *The derivative $dF_x : T_x\Sigma \to T_{(f(x),V(x))}(TM)$ is given by*

$$dF_x(u) = (df_x(u), (f^*\nabla)_u V(x)).$$

*Proof.* The hypersurface $\Sigma$ comes equipped with the induced pullback connection $f^*\nabla$. If $\gamma$ is a curve in $\Sigma$ starting at $x$ in the direction $u$ then

$$dF_x(u) = \frac{d}{dt}\, F(\gamma(t))|_{t=0} = \frac{d}{dt}\, (f(\gamma(t)), V(\gamma(t))|_{t=0} = (df_x(u), D_t(V \circ \gamma))(0))$$

where $D_t(V \circ \gamma)$ is the covariant derivative of the vector field $V$ along $\gamma$ with respect to the pullback connection. Since $V \circ \gamma$ is extendible,

$$D_t(V \circ \gamma) = (f^*\nabla)_{\gamma'} V.$$

Evaluating at $t = 0$ gives the result. □

If $M$ has a Riemannian metric and $\Sigma$ the pullback metric, then the special case when $V$ is orthogonal to $df(T\Sigma)$ deserves attention. If $V$ is a unit normal vector field on $\Sigma$, then the second fundamental form may be defined from the Gauss equation. The shape operator $B$ is then defined and the Weingarten equation relates these objects: (call $V = N$ now)

$$(f^*\nabla)_u N = -df_x(Bu).$$

In particular,

**Corollary.** *Suppose $f : \Sigma \to M$ is an isometric immersion and suppose $\Sigma$ has a unit normal vector field $N$. Then the map $F(x) = (f(x), N(x))$ gives a map $F : \Sigma \to TM$ and its derivative is*

$$dF_x(u) = (df_x(u), -df_x(Bu)).$$

## 4. Riemannian Manifolds

Suppose $M$ has a Riemannian metric $g$. Then there is a unique torsion free connection on $M$ such that is also metric, i.e.,

$$d\, g(X, Y) = g(\nabla X, Y) + g(X, \nabla Y).$$

4.1. **The Unit Tangent Bundle.** Inside $TM$ sits the unit tangent bundle $UM$ which consists of all tangent vectors of length 1. The unit tangent bundle is itself a bundle over $M$ with the same projection map. Since $UM$ lives inside $TM$ we know

$$T_{(p,v)}UM \leq H_{(p,v)}(TM) \oplus V_{(p,v)}(TM).$$

So take $(x, y)$ a tangent vector to $UM$ and a curve $\alpha(t) = (\gamma(t), V(t))$ in $UM$ with $\alpha'(0) = (x, y)$. This means $D_tV(0) = y$. From $1 = g(V, V)$ we compute

$$0 = \frac{d}{dt} g(V, V) = 2g(D_tV, V).$$

Hence $D_tV$ is orthogonal to $V$, and at $t = 0$ this gives $y \perp v$. Also, the vertical bundle of $UM$ is

$$V_{(p,v)}UM = \ker d\pi_{(p,v)} = T_{(p,v)}U_pM = v^\perp.$$

So we have

$$T_{(p,v)}UM \leq H_{(p,v)}(TM) \oplus v^\perp$$

This is actually an equality. Suppose $(x, y)$ is tangent to $TM$ at $(p, v)$ and that $y \perp v$. Let $(\gamma(t), V(t))$ be a curve in $TM$ whose initial velocity is $(x, y)$. Since $v \neq 0$, for $t$ in some small interval containing 0, we know $V(t) \neq 0$. We can define $W(t) = V(t)/|V(t)|$, and note that $(\gamma(t), W(t))$ is a path in $UM$. Moreover, the initial velocity to this curve is also $(x, y)$. To see this note that $W = g(V, V)^{-1/2}V$ and so

$$D_tW = -\frac{1}{2}g(V, V)^{-3/2} \cdot 2g(D_tV, V) \cdot V + g(V, V)^{-1/2}D_tV.$$

At $t = 0$ this gives $D_tW(0) = y$ since $D_tV(0) = y$ and since $y$ is orthogonal to $v$. Hence

**Lemma.** *The tangent space to $UM$ at $(p, v)$ is given by*

$$T_{(p,v)}UM = H_{(p,v)}(TM) \oplus v^\perp$$

*and the tangent space to $TM$ can be written as*

$$T_{(p,v)}TM = H_{(p,v)}TM \oplus v^\perp \oplus \langle v\rangle.$$

*where $\langle v\rangle$ is the span of $v$ in $T_pM$.*

4.2. **The Sasaki Metric.** Since both the horizontal space and vertical space are copies of $T_pM$, we can place a inner product on $T_{(p,v)}TM$ by using $g_p$ on each copy of $T_pM$ and making the horizontal and vertical spaces orthogonal to each other. The Sasaki Metric $\hat{g}$ is this metric, it is a Riemannian metric on $TM$ (and induces one on $UM$) and is given by

$$\hat{g}_{(p,v)}((x, y), (u, w)) = g_p(x, u) + g_p(y, w).$$

If $\alpha(t) = (\gamma(t), V(t))$ is a path in $TM$, then

$$|\alpha'|^2_{\hat{g}} = |\gamma'|^2_g + |D_tV|^2_g.$$

4.3. **Geodesics.** The Geodesic vector field is the horizontal vector field on the tangent bundle $G : TM \to T(TM)$ given by

$$G(p, v) = (v, 0).$$

If $\alpha(t) = (\gamma(t), V(t))$ is an integral curve of $G$, then

$$(\gamma'(t), D_tV(t)) = \alpha'(t) = G(\alpha(t)) = G(\gamma(t), V(t)) = (V(t), 0).$$

Hence, $V(t) = \gamma'(t)$ and $D_t\gamma' = D_tV = 0$.

**Lemma.** *The (projections of the) integral curves of $G$ are called geodesics and they are given by*

$$\tilde{\gamma}(t) = (\gamma(t), \gamma'(t)) \text{ such that } D_t\gamma' = 0.$$

If $\gamma$ is a geodesic then note that $\partial_t|\gamma'|^2 = 2g(D_t\gamma', \gamma') = 0$ so that $|\gamma'|$ is constant. We can assume $\gamma$ has been parametrized so that $|\gamma'(0)| = 1$, in which case we see the integral curves of $G$ are paths in $UM$. Indeed we can consider the geodesics vector field as a unit vector field on $TM$ and its Sasaki norm is $|G|_{\hat{g}} = 1$.

4.4. **The exponential map.** The exponential map on $M$ is the map $\exp : TM \to M$ by

$$\exp(p,v) = \exp_p(v) = \gamma(1),$$

where $\gamma$ is the unique geodesic starting at $p$ in the direction $v$.

We can compute $d\exp_{(p,v)} : T_{(p,v)}TM \to T_{\exp_p(v)}M$ using Jacobi fields.

**Theorem.** *The derivative of the exponential map is given by*

$$d\exp_{(p,v)}(x,y) = J(1)$$

*where $J(t)$ is the unique Jacobi field along $\exp_p(tv)$ with $J(0) = x$ and $D_tJ(0) = y$.*

*Proof.* Take a path in $TM$ by $\alpha(s) = (\gamma(s), V(s))$ such that $\alpha'(0) = (\gamma'(0), D_sV(0)) = (x,y)$. Consider the variation through geodesics given by

$$\Gamma(s,t) = \exp_{\gamma(s)}(tV(s)).$$

Then there exists a unique Jacobi field $J(t) = \partial_s\Gamma(0,t)$ along $\Gamma(0,t) = \exp_p(tv)$ such that $J(0) = x$ and $D_tJ(0) = y$. We can then compute

$$\begin{aligned} d\exp_{(p,v)}(x,y) &= \frac{d}{ds}\,\exp(\alpha(s))\big|_{s=0} \\ &= \frac{d}{ds}\,\exp_{\gamma(s)}(1\cdot V(s))\Big|_{s=0} \\ &= \frac{d}{ds}\,\Gamma(s,1)\big|_{s=0} \\ &= \partial_s\Gamma(0,1) \\ &= J(1). \end{aligned}$$

□

4.5. **The Geodesic Flow.** The flow $g^t$ of the geodesic vector field $G$ is called the geodesic flow. So, $g^t : TM \to TM$ by $g^t(p,v) = (\gamma(t), \gamma'(t))$ for $\gamma$ the unique geodesic starting at $p$ in the direction $v$. Equivalently

$$g^t(p,v) = (\exp_p(tv), \partial_t \exp_p(tv)).$$

**Theorem.** *The derivative of the geodesic flow $g^t : TM \to TM$ is given by*

$$d(g^t)_{(p,v)}(x,y) = (J(t), D_tJ(t)).$$

*where $J(t)$ is the unique Jacobi field along $\exp_p(tv)$ with $J(0) = x$ and $D_tJ(0) = y$.*

*Proof.* Take $\alpha(s) = (\gamma(s), V(s))$ a curve in $TM$ with $\gamma'(0) = x$ and $D_sV(0) = y$ and again form the variation through geodesics $\Gamma(s,t) = \exp_{\gamma(s)}(tV(s))$ which has Jacobi field $J(t) = \partial_s\Gamma(0,t)$ satisfying $J(0) = x$ and $D_tJ(0) = y$. We can compute

$$\begin{aligned} d(g^t)_{(p,v)}(x,y) = \frac{d}{ds}\,g^t(\alpha(s))\big|_{s=0} &= \frac{d}{ds}\,(\exp_{\gamma(s)}(tV(s)), \partial_t \exp_{\gamma(s)}(tV(s)))\Big|_{s=0} \\ &= (J(t), D_s\partial_t \exp_{\gamma(s)}(tV(s)))\Big|_{s=0}, \end{aligned}$$

where the covariant derivative is along $\exp_{\gamma(s)}(tV(s))$. Now the symmetry lemma gives

$$\begin{aligned} D_s\partial_t \exp_{\gamma(s)}(tV(s))\Big|_{s=0} &= D_s\partial_t\Gamma(s,t)|_{s=0} \\ &= D_t\partial_s\Gamma(s,t)|_{s=0} \\ &= D_t\partial_s\Gamma(0,t) \\ &= D_tJ(t), \end{aligned}$$

as claimed. □

Note that $g^0(p,v) = (p,v)$ and indeed we have $d(g^0)_{(p,v)}(x,y) = (J(0), D_tJ(0)) = (x,y)$.

4.6. **Parallel Surfaces.** Now assume again that $F : \Sigma \to UM$ by $F(x) = (f(x), N(x))$ is the lift of a hypersurface $f : \Sigma \to M$ to the unit tangent bundle. We can then consider the parallel surface $\Sigma_t$ a distance $t$ away from $\Sigma$ in $UM$ by taking $F^t = g^t \circ F : \Sigma \to M$.

**Theorem.** *Let $F^t = g^t \circ F : \Sigma \to UM$. Then the derivative is*

$$d(F^t)_x(u) = (J(t), D_tJ(t))$$

*where $J(t)$ is the unique Jacobi field along $\exp_{f(x)}(tN(x))$ with $J(0) = df_x(u)$ and $D_tJ(0) = -df_x(Bu))$ for $B$ the shape operator of $\Sigma$ in $M$.*

*Proof.* Recall that $dF_x(u) = (df_x(u), -df_x(Bu))$. Then since $d(F^t)_x(u) = d(g^t)_{F(x)} \circ dF_x(u)$ we get from the result from the above theorem. □

If we now consider $f^t = \pi \circ F^t : \Sigma \to M$ then we get a parallel copy of $\Sigma$ a distance $t$ away. This copy has its own induced metric $I_t = (f^t)^*g$ and $I_0 = I = f^*g$. We can compute the derivative of this family $I_t$ with respect to $t$.

**Theorem.** *Let $f : \Sigma \to M$ be an immersed hypersurface and let $N$ be a unit normal vector field on $\Sigma$. Then for the induced metric on the parallel surface a distance $t$ away from $\Sigma$ in the direction $N$ we have*

$$\frac{d}{dt}\, I_t\big|_{t=0} = -2I\!I.$$

*where $I\!I$ is the second fundamental form of the immersion $f$ with respect to $N$.*

*Proof.* From the previous theorem we have that $d(f^t)_x(u) = J(t)$ where $J$ is the unique Jacobi field along $\exp_{f(x)}(tN(x))$ with $J(0) = df_x(u)$ and $D_tJ(0) = -df_x(Bu)$. Let $d(f^t)_x(w) = W(t)$ with similar conditions on the Jacobi field $W$. Then

$$I_t(u,w) = g(d(f^t)_x(u), d(f^t)_x(w)) = g(J(t), W(t)).$$

Taking the derivative with respect to time gives

$$\frac{d}{dt}I_t(u,w) = \frac{d}{dt}g(J(t),W(t)) = g(D_tJ,W) + g(J,D_tW).$$

At $t = 0$ this becomes

$$\begin{aligned}\frac{d}{dt}\, I_t(u,w)|_{t=0} &= g(-df_x(Bu), df_x(w)) + g(df_x(w), -df_x(Bw))\\ &= -I(Bu,w) - I(u,Bw)\\ &= -2I(Bu,w)\\ &= -2I\!I(u,w).\end{aligned}$$

□

The second derivative can also be computed by using the Jacobi equation

**Theorem.** *Let $f : \Sigma \to M$ be an immersed hypersurface and let $N$ be a unit normal vector field on $\Sigma$. Then the induced metric on the parallel surface a distance $t$ away from $\Sigma$ in the direction $N$ we have*

$$\frac{d^2}{dt^2}\, I_t|_{t=0} = 2I\!I\!I(u,w) - 2\widetilde{Rm}(df_x(u), N(x), N(x), df_x(w)).$$

*where $I\!I\!I$ is the third fundamental form of the immersion $f$ with respect to $N$ and $\widetilde{Rm}$ is the Riemann curvature tensor of $g$.*

The unit tangent bundle has the Sasaki metric and so given a unit normal immersion $F : \Sigma \to UM$ by $F(x) = (f(x), N(x))$ we can computed the induced metric on $\Sigma$.

**Theorem.** *Let $F : \Sigma \to UM$ by $F(x) = (f(x), N(x))$ be a unit normal immersion into the unit tangent bundle of a manifold with metric $g$. Let $\hat{g}$ be the Sasaki metric on $UM$ and call the induced metric $\hat{I} = F^*\hat{g}$. Then*

$$\hat{I} = I + I\!I\!I$$

*where $I$ is the induced metric $f^*g$ on $\Sigma$ and $I\!I\!I$ is the third fundamental form.*

*Proof.* Recall the derivative of $F$ is $dF_x(u) = (df_x(u), -df_x(Bu))$. Then

$$\begin{aligned}F^*\hat{g}(u,w) &= \hat{g}(dF_x(u), dF_x(w))\\ &= \hat{g}((df_x(u), -df_x(Bu)), (df_x(w), -df_x(Bw)))\\ &= g(df_x(u), df_x(w)) + g(-df_x(Bu), -df_x(Bw))\\ &= I(u,w) + I(Bu,Bw)\\ &= I(u,w) + I\!I\!I(u,w).\end{aligned}$$

□

## 5. Hyperbolic Space

Minkowski space $\mathbb{R}^{n,1}$ is just $\mathbb{R}^{n+1}$ with the Minkowski metric

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1},$$

a scalar product of signature $(n, 1)$. Hyperbolic space sits inside of Minkowski space as the upper portion of the hyperbola of two sheets:

$$\mathbb{H}^n = \{p \in \mathbb{R}^{n,1} \mid \langle p, p\rangle = -1, \text{ and } p_{n+1} > 0\}.$$

De Sitter space is the unit sphere in the Minkowski metric:

$$dS^n = \{v \in \mathbb{R}^{n,1} \mid \langle v, v\rangle = 1\}.$$

Since both of these spaces are level sets of $\langle \cdot, \cdot \rangle$, their tangent spaces at a point can be identified with the orthogonal complements

$$T_p\mathbb{H}^n = p^\perp \quad \text{and} \quad T_v dS^n = v^\perp.$$

It can be seen that hyperbolic space is a space-like submanifold of Minkowski space, meaning the restriction of the Minkowski metric to $\mathbb{H}^n$ is positive definite. De Sitter space is a Lorentzian submanifold of de Sitter space.

Note that if $v$ is a unit tangent vector to $p$ in hyperbolic space, then $v \in U_p\mathbb{H}^n \leq T_p\mathbb{H}^n \leq \mathbb{R}^{n,1}$ and $\langle v, v \rangle = 1$. Consequently, $v \in dS^n$. Moreover, since $\langle p, v \rangle = \langle v, p \rangle = 0$, the point $p \in \mathbb{R}^{n,1}$ is actually tangent to $dS^n$ at $v$. In summary, we have a map

$$U\mathbb{H}^n \to TdS^n \quad (p, v) \mapsto (v, p).$$

5.1. **Geometry of hyperbolic and de Sitter Space.** The Levi-Civita connection of $\mathbb{R}^{n,1}$ is the same as that of $\mathbb{R}^{n+1}$ with the Euclidean metric. So, the geodesics of Minkowski space are the straight lines; the geodesic through $p$ in the direction $v$ is $\gamma(t) = p + tv$. For the coming computations, it will be useful to know the second fundamental form of hyperbolic space as a submanifold of Minkowski space. Note, though, that since $\mathbb{H}^n$ is space-like in Minkowski space, the metric restricted to its normal bundle is negative definite. This means any unit normal vector field will have length $-1$. Since $T_p\mathbb{H}^n = p^\perp$, it's quick to see that $p$ itself is a vector in $\mathbb{R}^{n,1}$ orthogonal to its tangent space. So we can define a unit normal vector field to hyperbolic space by $N(p) = p$, which is just the inclusion. Similarly, the inclusion of de Sitter space into $\mathbb{R}^{n,1}$ provide a unit normal vector field for $dS^n$.

Now, we can compute the second fundamental form of hyperbolic space by flowing it in its normal direction using a variant of Theorem above. However, in that Theorem we used the formula $(f^*\nabla)_u N = -df_p(Bu)$, which is for Riemannian submanifolds of Riemannian manifolds. In the Riemannian submanifold of Lorentzian manifolds case, the correct formula is $(f^*\nabla)_u N = df_p(Bu)$. This means that in this case, $\frac{d}{dt}\, I_t|_{t=0} = 2\mathit{II}$.

**Lemma.** *Let $\mathit{II}^{\mathbb{H}}$ be the second fundamental form of hyperbolic space in $\mathbb{R}^{n,1}$, then*

$$\mathit{II}^{\mathbb{H}}(u, w) = \langle u, w \rangle$$

*Proof.* We use $\frac{d}{dt}\, I_t|_{t=0} = 2\mathit{II}$. Let $g^t$ be the geodesic flow of $\mathbb{R}^{n,1}$, then since the geodesics are straight lines, we have that $g^t(p, N(p)) = p + tN(p) = (1+t)p$. Consequently, $d(g^t)_p(v) = (1+t)v$.

$$I_t^{\mathbb{H}}(u, w) = \frac{1}{2}\langle (1+t)u, (1+t)w \rangle = \frac{1}{2}(1+t)^2 \langle u, w \rangle.$$

Taking the derivative at $t = 0$ then gives the result. □

For de Sitter space, the original $\frac{d}{dt}\, I_t|_{t=0} = -2\mathit{II}$ holds. A similar computation then gives $\mathit{II}^{dS}(u, w) = -\langle u, w \rangle$.

**Proposition.** *The Gauss formula for hyperbolic space in Minkowski space is*

$$\bar{\nabla}_X Y = \nabla_X^{\mathbb{H}} Y + \langle X, Y \rangle N$$

*and for de Sitter space is*

$$\bar{\nabla}_X Y = \nabla_X^{dS} Y - \langle X, Y \rangle N.$$

*where $\bar{\nabla}$ is the flat Levi-Civita connection of $\mathbb{R}^{n,1}$ (and $\mathbb{R}^{n+1}$).*

Using the Gauss formula in $\mathbb{H}^n$, the geodesic equation becomes $\gamma'' = |\gamma'|^2\gamma$, an equation in $\mathbb{R}^{n,1}$. If we restrict to the unit tangent bundle, then the equation is $\gamma'' = \gamma$ and this has solutions

$$\gamma(t) = \cosh(t)\gamma(0) + \sinh(t)\gamma'(0).$$

Hence, the geodesic flow in hyperbolic space is

$$g^t(p, v) = \cosh(t)p + \sinh(t)v.$$

We can now compute the derivative of the geodesic flow on hyperbolic space.

**Theorem.** *Let $g^t : U\mathbb{H}^n \to U\mathbb{H}^n$ be the geodesic flow. The derivative is given by*

$$d(g^t)_{(p,v)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh(t)x + \sinh(t)y + \sinh(t)\langle x, v\rangle p \\ \sinh(t)x + \cosh(t)y - \sinh(t)\langle x, v\rangle v \end{pmatrix},$$

*with respect to the splitting $T(TM) = H \oplus V$.*

*Proof.* We know from Theorem that the derivative of the geodesic flow related to Jacobi fields. In order to verify a proposed Jacobi field does in fact satisfy the Jacobi equation we would need to compute the Riemann curvature endomorphism of hyperbolic space. To avoid doing this, we compute the derivative of the geodesic flow directly, as it is not too complicated in this model.

Let $\alpha(s) = (\gamma(s), V(s))$ be a curve in $U\mathbb{H}^n$ with $\alpha(0) = (p, v)$ and $\alpha'(0) = (x, y)$. Recall this means that $\gamma'(0) = x$ and $D_sV(0) = y$. Recall also that $y \perp v$. Then we compute

$$\begin{aligned} d(g^t)_{(p,v)}(x, y) &= \frac{d}{ds}\, g^t(\alpha(s))\big|_{s=0} \\ &= \frac{d}{ds}\, (\cosh(t)\gamma(s) + \sinh(t)V(s), \sinh(t)\gamma(s) + \cosh(t)V(s))\big|_{s=0} \\ &= (\cosh(t)x + \sinh(t)V'(0), D_s(\sinh(t)\gamma(s) + \cosh(t)V(s))(0)), \end{aligned}$$

where the covariant derivative is along $\cosh(t)\gamma(s) + \sinh(t)V(s)$ and $V'(0)$ is the derivative of $V : (-1, 1) \to \mathbb{R}^{n,1}$, which is equal to $\bar{D}_sV(0)$.

Using the Gauss formula we have

$$V'(0) = \bar{D}_sV(0) = D_sV(0) + \langle\gamma'(0), V(0)\rangle\gamma(0) = y + \langle x, v\rangle p,$$

and so the first term of the derivative is $\cosh(t)x + \sinh(t)y + \sinh(t)\langle x, v\rangle p$. For the second term, let $\beta(s) = \cosh(t)\gamma(s) + \sinh(t)V(s)$ and $\delta(s) = \sinh(t)\gamma(s) + \cosh(t)V(s)$, so that the covariant derivative we need to compute is $D_s\delta$ along $\beta$. Then, again using the Gauss formula, we get $D_s\delta = \bar{D}_s\delta - \langle\beta', \delta\rangle\beta$. Each term is

$$\begin{aligned} \bar{D}_s\delta &= \delta'(s) = \sinh(t)\gamma'(s) + \cosh(t)V'(s), \\ \langle\beta', \delta\rangle &= \langle\cosh(t)\gamma'(s) + \sinh(t)V'(s), \sinh(t)\gamma(s) + \cosh(t)V(S)\rangle, \\ \beta(s) &= \cosh(t)\gamma(s) + \sinh(t)V(s). \end{aligned}$$

At $s = 0$ these become

$$\begin{aligned} \bar{D}_s\delta(0) &= \delta'(0) = \sinh(t)x + \cosh(t)V'(0), \\ \langle\beta', \delta\rangle &= \langle\cosh(t)x + \sinh(t)V'(0), \sinh(t)p + \cosh(t)v\rangle, \\ \beta(s) &= \cosh(t)p + \sinh(t)v. \end{aligned}$$

Using $V'(0) = y + \langle x, v\rangle p$, the fact that $v, x, y \perp p$ and $y \perp v$, plugging everything into the Gauss formula and then simplifying gives the second term and the result. $\square$

Notice that at $t = 0$, the geodesic flow is $g^0 = Id$ and we do get the correct $d(g^0)_{(p,v)}(x, y) = (x + 0 + 0, 0 + y - 0) = (x, y)$.

In a similar way, we can compute the derivative of the map $U\mathbb{H}^n \to TdS^n$ where $(p, v) \mapsto (v, p)$.

**Theorem.** *Let $i : U\mathbb{H}^n \to TdS^n$ be given by $i(p, v) = (v, p)$. Then with respect to the splitting of the double tangent bundle of both $\mathbb{H}^n$ and $dS^n$ into horizontal and vertical bundles (via the metrics induced from Minkowski space), we can write the derivative as*

$$di_{(p,v)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + \langle x, v\rangle p \\ x - \langle x, v\rangle v \end{pmatrix}$$

*Proof.* Let $\alpha(t) = (\gamma(t), V(t))$ be a curve in $U\mathbb{H}^n$ with $\alpha(0) = (p, v)$ and $\alpha'(0) = (x, y)$. Recall this means that $\gamma'(0) = x$ and $D_sV(0) = y$. Recall also that $y \perp v$. Then we compute

$$\begin{aligned} di_{(p,v)}(x, y) &= \frac{d}{dt}\, i(\alpha(t))|_{t=0} \\ &= \frac{d}{dt}\, (V(t), \gamma(t))|_{s=0} \\ &= (V'(0), D_t^{dS}\gamma(0)), \end{aligned}$$

where $D^{dS}$ is the covariant derivative with respect to de Sitter space and we think of $\gamma(t)$ as a vector field along $V(t)$ in $dS^n$. Recall from the proof of the previous Theorem that $V'(0) = \bar{D}_tV(0) = y + \langle x, y\rangle p$. Then, using de Sitter space's Guass formula we have

$$\begin{aligned} D_t^{dS}\gamma(0) &= \bar{D}_t\gamma(0) + \langle V'(0), \gamma'(0)\rangle V(0) \\ &= \gamma'(0) + \langle y + \langle x, v\rangle p, p\rangle v \\ &= x - \langle x, v\rangle v \end{aligned}$$

$\square$

If we consider $TdS^n$ with its Sasaki metric, then we can compute the metric on $U\mathbb{H}^n$ induced from $i$.

**Lemma.** *Let $\hat{g}_{dS}$ be the Sasaki metric on $TdS^n$. Then*

$$\begin{aligned} i^*(\hat{g}_{dS})((x, y), (u, w)) &= \langle x, y\rangle - 2\langle x, v\rangle\langle u, v\rangle + \langle y, w\rangle \\ &= \hat{g}_{\mathbb{H}}((x, y), (u, w)) - 2\langle x, v\rangle\langle u, v\rangle, \end{aligned}$$

*where $\hat{g}_{\mathbb{H}}$ is the Sasaki metric on $U\mathbb{H}^n$.*

5.2. **Surfaces in Hyperbolic Space.** Suppose we have $f : \Sigma \to \mathbb{H}^n$ and have a unit normal lift $F : \Sigma \to U\mathbb{H}^n$ given by $F(x) = (f(x), N(x))$. Then composing $F$ with $i : U\mathbb{H}^n \to TdS^n$ we get a dual surface in de Sitter space $F^* = i \circ F : \Sigma \to TdS^n$ and $f^* = \pi \circ i \circ F : \Sigma \to dS^n$.

**Lemma.** *The derivative of $F^*$ is given by*

$$dF_x^*(u) = \begin{pmatrix} -df_x(Bu) \\ df_x(u) \end{pmatrix},$$

*and the derivative of* $f^*$ *is*

$$df^*_x(u) = -df_x(Bu).$$

*Proof.* Since $F^*$ is the composition $i \circ F$ we have

$$dF^*_x(u) = di_{(f(x),N(x))} \circ dF_x(u) = di_{(f(x),N(x))}(df_x(u), -df_x(Bu)).$$

Then

$$di_{(f(x),N(x))} \begin{pmatrix} df_x(u) \\ -df_x(Bu) \end{pmatrix} = \begin{pmatrix} -df_x(Bu) + \langle df_x(u), N(x)\rangle p \\ df_x(u) - \langle df_x(u), N(x)\rangle N(x) \end{pmatrix}.$$

The result then follows from $df_x \perp N(x)$. $\square$

The induced metrics $\hat{I}^* = (F^*)^*\hat{g}_{dS}$ and $I^* = (f^*)^* g_{dS}$ can now be computed

**Theorem.** *Let* $f : \Sigma \to \mathbb{H}^n$ *be a map and* $F : \Sigma \to U\mathbb{H}^n$ *via* $F(x) = (f(x), N(x))$ *be a unit normal immersion. Then the metrics* $I^*$ *and* $\hat{I}^*$ *on* $\Sigma$ *obtained by pulling back the metric on* $dS^n$ *or the Sasaki metric on* $TdS^n$*, respectively, are given by*

$$\hat{I}^* = I + I\!I\!I,$$

*and*

$$I^* = I\!I\!I$$

Compare this with Theorem. We see that the induced metric $\hat{I}$ from the unit normal lift to $U\mathbb{H}^n$ with its Sasaki metric is the same as the dual induced metric $\hat{I}^*$. The induced metric on the dual surface $\Sigma^* = f^* : \Sigma \to dS^n$ is given as the third fundamental form of $\Sigma$. We can use this to determine when $\Sigma^*$ is immersed.

**Proposition.** *The dual surface* $\Sigma^*$ *is immersed in* $dS^n$ *provided* $\Sigma$ *is strictly convex in* $\mathbb{H}^n$*.*

(CHECK CORRECT TERMINOLOGY)

*Proof.* The map $f^* : \Sigma \to dS^n$ is an immersion when the induced metric $I^* = I\!I\!I$ is positive definite (since $df^*$ takes values in space-like vectors). Now, since $I\!I\!I(u, w) = I(Bu, Bw) = I(B^2u, w)$, the third fundamental form will be positive definite if $B^2$ has strictly positive eigenvalues. As the eigenvalues of $B^2$ are the square of the eigenvalues of $B$ (which are real), the third fundamental form will be positive definite when the eigenvalues of $B$ are non-zero. And this happens whenever $I\!I$ is definite, meaning $\Sigma$ is strictly convex (CHECK IF NEED POSITIVE TOO NOT JUST POSITIVE DEFINITE). $\square$

If $F$ is a unit normal lift of $f : \Sigma \to \mathbb{H}^n$, then composing with the geodesic flow gives $F_t = g^t \circ F$ to get parallel surfaces $\Sigma_t = f_t = \pi \circ F_t$ in hyperbolic space a distance $t$ away from $\Sigma$. Define $I_t = (f_t)^* g_{\mathbb{H}}$ as above. We can compute this induced metric in terms of the geometry on $\Sigma$. Indeed, the derivative of $F_t = g^t \circ F$

can be computed via the chain rule as

$$\begin{aligned}
d(F_t)_x(u) &= d(g^t)_{(f(x),N(x))} \circ dF_x(u) \\
&= d(g^t)_{(f(x),N(x))} \begin{pmatrix} df_x(u) \\ -df_x(Bu) \end{pmatrix} \\
&= \begin{pmatrix} \cosh(t)df_x(u) - \sinh(t)df_x(Bu) + \sinh(t)\langle df_x(u), N(x)\rangle p \\ \sinh(t)df_x(u) - \cosh(t)df_x(Bu) - \sinh(t)\langle df_x(u), N(x)\rangle N(x) \end{pmatrix} \\
&= \begin{pmatrix} \cosh(t)df_x(u) - \sinh(t)df_x(Bu) \\ \sinh(t)df_x(u) - \cosh(t)df_x(Bu) \end{pmatrix} \\
&= \begin{pmatrix} df_x(\cosh(t)Id - \sinh(t)B)u \\ df_x(\sinh(t)Id - \cosh(t)B)u \end{pmatrix}
\end{aligned}$$

Let $A_t = \cosh(t)Id - \sinh(t)B$, then the derivative of $f_t$ is $d(f_t)_x(u) = df_x(A_t(u))$ or

$$d(f_t)_x = df_x \circ A_t.$$

Consequently,

**Theorem.** *Let $f : \Sigma \to \mathbb{H}^n$ be an immersion with unit normal vector field $N$ and shape operator $B$. Define $f_t(x) = \pi \circ g^t(f(x), N(x))$, then the image of $f_t$ is a surface a distance $t$ away from $\Sigma$ and has in induced metric*

$$I_t(u, w) = I((\cosh(t)Id - \sinh(t)B)u, (\cosh(t)Id - \sinh(t)B)w)$$

*and shape operator*

$$B_t = -(\cosh(t)Id - \sinh(t)B)^{-1}(\sinh(t)Id + \cosh(t)B).$$

*Proof.* The first claim follows immediately from the definition of a pullback metric along with the formula for $d(f_t)_x$. For the moment, let $f : \Sigma \to \mathbb{H}^n$ be any immersion with unit normal vector field $N^\Sigma$. Then the Gauss formula says

$$(f^*\nabla^{\mathbb{H}})_X df(Y) = df(\nabla^\Sigma_X Y) + I\!I^\Sigma(X, Y)N^\Sigma.$$

From the Gauss equation for hyperbolic space in Minkowski space

$$\bar{\nabla}_X Y = \nabla^{\mathbb{H}}_X Y + \langle X, Y\rangle N^{\mathbb{H}},$$

we can take the pullback to get

$$(f^*\bar{\nabla})_X Y = (f^*\nabla^{\mathbb{H}})_X Y + \langle X, Y\rangle(N^{\mathbb{H}} \circ f)$$

where $Y$ is now a section of $f^*T\mathbb{H}^n$. In the case where $Y = N^\Sigma$, we see

$$(f^*\nabla^{\mathbb{H}})_u N^\Sigma = (f^*\bar{\nabla})_u N^\Sigma - \langle u, N^\Sigma\rangle f = (f^*\bar{\nabla})_u N^\Sigma$$

since $N^\Sigma$ is orthogonal to $u$. The Weingarten equation then says $df_x \circ Bu = -(f^*\bar{\nabla})_u N^\Sigma$. Choose a path $\gamma$ in $\Sigma$ such that $\gamma(0) = x$ and $\gamma'(0) = u$. Then the Weingarten equation becomes

$$df_x \circ Bu = -\bar{D}_t N(0),$$

the covariant derivative being taken along $\gamma$.

Now, specialize to $f_t : \Sigma \to \mathbb{H}^n$ with unit normal $N_t = \partial_t(\cosh(t)f(x) + \sinh(t)N(x)) = \cosh(t)N(x) + \sinh(t)f(x)$. From the above, we have for a similar path $\gamma(s)$

$$d(f_t)_x \circ B_t u = -(f_t^*\nabla^{\mathbb{H}})_u N_t = -\bar{D}_s N_t(0).$$

It is straight forward to compute

$$\begin{aligned}\bar{D}_s N_t &= \cosh(t)\bar{D}_s N(0) + \sinh(t)\bar{D}_s f(\gamma(s))(0)\\ &= \cosh(t)(df_x \circ Bu) + \sinh(t) df_x(u)\\ &= df_x(\sinh(t)Id + \cosh(t)B)u.\end{aligned}$$

Recall that $d(f_t)_x = df_x \circ A_t$. We have

$$df_x \circ A_t \circ B_t u = -df_x(\sinh(t)Id + \cosh(t)B)u.$$

Since $f$ is an immersion, $df_x$ is injective and

$$A_t \circ B_t u = -(\sinh(t)Id + \cosh(t)B)u \implies B_t u = -A_t^{-1}(\sinh(t)Id + \cosh(t)B)u,$$

and the result follows. □